Putting Some Harmony Into the Harmonic Series

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The Harmonic Series

- The harmonic series defined

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- The harmonic series not only shows up in mathematics, but also in architecture, music, and physics
- The harmonic series was first shown to diverge by Oresme (14th century), but proofs by Mengoli, Johann Bernoulli, and Jakob Bernoulli are most well known.

Harmonic Series Diverges

Proof (By Contradiction) Assume not. That is, assume that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges and its sum is L.

$$
\begin{aligned}
L &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\
&= \left(1 + \frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8}\right) + \cdots \\
&> \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots \\
&= L
\end{aligned}
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Note: This is a contradiction since $L \not> L$.

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Note: This is a contradiction since $L \not> L$.

Delete All Terms Except Reciprocals of Prime Numbers

The series is:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \cdots$$

- There is an infinite number of primes, so there is an infinite number of terms.
- We have also removed an infinite number of terms.
- Does this series converge?
- In 1737, Euler showed that this series diverges.
- For a proof, see Dunham, 1999, page 76.

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Delete All Terms According to a Pattern

Consider the set $S = \{1, 3, 6, 10, 15, 21, 28, 36, 45, \ldots\}$ formed by removing $\{1, 2, 3, 4, \ldots\}$ members in the original sequence of natural numbers

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, \ldots\}$$

.

$$\sum_{k \in S} \frac{1}{k} = 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \frac{1}{21} \cdots$$

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Delete All Terms According to a Pattern

$$\sum_{k\in S}\frac{1}{k}=1+\frac{1}{3}+\frac{1}{3\times 2}+\frac{1}{5\times 2}+\frac{1}{5\times 3}+\frac{1}{7\times 3}+\frac{1}{7\times 4}+\cdots$$

$$=1+\frac{1}{3}\left(\frac{2+1}{2}\right)+\frac{1}{5}\left(\frac{3+2}{2\times 3}\right)+\frac{1}{7}\left(\frac{4+3}{3\times 4}\right)+\cdots$$

$$=1+\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\frac{1}{4\times 5}+\cdots$$

$$=1+\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{k-1}-\frac{1}{k}\right)+\cdots$$

$$=\lim_{k\to+\infty}\left(2-\frac{1}{k}\right)=2$$

This series converges to 2!

Delete All Terms According to a Pattern

$$\begin{aligned}\sum_{k \in S} \frac{1}{k} &= 1 + \frac{1}{3} + \frac{1}{3 \times 2} + \frac{1}{5 \times 2} + \frac{1}{5 \times 3} + \frac{1}{7 \times 3} + \frac{1}{7 \times 4} + \cdots \\ &= 1 + \frac{1}{3}\left(\frac{2+1}{2}\right) + \frac{1}{5}\left(\frac{3+2}{2 \times 3}\right) + \frac{1}{7}\left(\frac{4+3}{3 \times 4}\right) + \cdots \\ &= 1 + \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \cdots \\ &= 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k}\right) + \cdots\end{aligned}$$

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Delete All Terms That Include A Particular Digit, Say 9

The series is:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{8}+\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{18}+\frac{1}{20}$$
$$+\cdots+\frac{1}{88}+\frac{1}{100}+\cdots+\frac{1}{108}+\frac{1}{110}+\cdots.$$

- Does this series converge or diverge?
- Proven to converge in 1914 by Kempner.
- Proof is by induction.
- The fact that this series converges tells us something about the density of the digits in numbers.

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Grouping the 9-less Series

Grouping the terms yields:

$$\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{8}\right)+\left(\frac{1}{10}+\cdots\frac{1}{88}\right)$$

$$+\left(\frac{1}{100}+\cdots+\frac{1}{888}\right)+\cdots.$$

If we let a_n represent the sum of the nth group of terms, then the series can be written as

$$a_1+a_2+a_3+\cdots.$$

- Observe that the first and greatest fraction in a_n is $1/10^{n-1}$.
- Claim - There are fewer than 9^n terms in a_n.
- This implies that the value of $a_n < 9^n/10^{n-1}$.

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Proof of Convergence (by induction - by Honsberger, 1976, page 98)

Claim: The number of terms in each group of a_n is bounded by 9^n.

- The number of terms in a_1 is $8 < 9^1$. That is, $(1, 1/2, 1/3, ..., 1/8)$.
- The number of terms in a_2 is $72 < 9^2$.
- Induction Hypothesis. Assume that the number of terms in a_k is less than 9^k for $k = 1, 2, 3, ..., n$.
- We will use this assumption to deduce that the number of terms in a_{n+1} is less than 9^{n+1}
- The group a_{n+1} contains $1/10^n$ and all fractions not deleted between $1/10^n$ and $1/10^{n+1}$

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Proof Continued 2

- All numerators are equal to 1 so let's look at the denominators and break up the range as follows:

$$\underbrace{10^n, \ldots, 2 \cdot 10^n, \ldots, 3 \cdot 10^n, \ldots, 8 \cdot 10^n, \ldots, 9 \cdot 10^n, \ldots, 10^{n+1}}_{\text{the } a_{n+1} \text{range}}$$

- All the numbers in the last section ($9 \cdot 10^n$ to 10^{n+1}) begin with 9, so all corresponding denominators would have been deleted.
- We need only to count the number of denominators in the first 8 sections, from 10^n up to $9 \cdot 10^n$.
- Each of these sections contain exactly the same number of terms as the number of terms included in the initial range from 0 to 10^n.

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- To see why this is true note that if a denominator was deleted in an earlier grouping, it will be deleted if a new digit is appended to it.
- For example, if the number with digits $b_1 b_2 \ldots b_k$ contains the digit 9, then certainly $3b_1 b_2 \ldots b_k$ also contains a 9.
- This implies that the number of fractions in a_{n+1} is

$$< 8(9 + 9^2 + \cdots + 9^n) = 8 \cdot \frac{9(9^n - 1)}{9 - 1} = 9^{n+1} - 9 < 9^{n+1}.$$

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Back to the series

- Since the largest fraction in each grouping is the first term, $1/10^{n-1}$, we can bound a_n by the product of the number of terms and the largest fraction. That is, $a_n < 9^n/10^{n-1}$.
- This implies that the sum of our "9-less" series is bounded by a geometric series. That is,

$$a_1 + a_2 + a_3 \cdots < \sum_{n=1}^{\infty} \frac{9^n}{10^{n-1}} = \frac{9}{1 - \frac{9}{10}} = 90.$$

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Other Results

- 1914 Kempner's Result
- In 1979 Ballie calculated (to 20 decimal places) the sum of the 9-less series.

$$22.92067661926415034816$$

- 1916 Irwin extended the result. "If we strike out from the harmonic series those terms whose denominators contain the digit 9 at least a times, and, at the same time, the digit 8 at least b times, the digit 7, at least c times, and so on, to the digit 0 at least j times $(a, b, c, \ldots, j)$ being any given integers, the series so obtained will converge."

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- In 1978 Wadhwa showed that a series thinned in the form of

$$\sum_{n \in S} \frac{1}{n^{\alpha}},$$

where S is the set of all positive integers that have been thinned as in the "9-less" series, converges, provided that $\alpha > \log_{10} 9 > 0.95$.

  i. Proof parallels proof given above
  ii. Note that all p-series converge where $\alpha > 1$.
  iii. Thus, these types of thinning processes only help us with p-series where $\log_{10} 9 < \alpha \leq 1$.

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Amazing Consequence of This Result

- Let's Look at the following ratio:

$$\frac{\text{number of natural numbers } < 10^{n+1} \text{ NOT containing 9}}{\text{number of natural numbers } < 10^{n+1} \text{ that contain 9}}$$

- We quantify this with our other bound as follows:

$$R < \frac{\frac{9(9^n-1)}{9-1}}{10^{n+1}-1-\frac{9(9^n-1)}{9-1}} = \frac{9(9^n-1)}{8(10^{n+1}-1)-9(9^n-1)}$$

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What Does This Imply?

- As $n \to \infty$ the ratio goes to zero.
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- We can also say the same for all of the other digits.
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