Points on a Circle with Integer Distances with Respect to an Equilateral Triangle

Solution to PME Journal Problem 1245 by Stanley Rabinowitz

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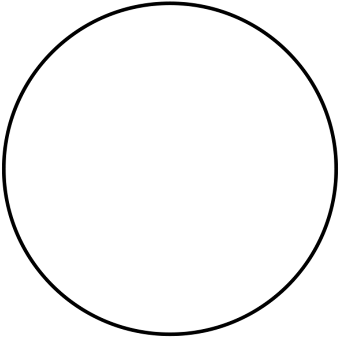
Advised by Dr. Jacek Fabrykowski

Let ABC be an equilateral triangle with edge length c inscribed in a circle. Let N be a point on minor arc $\overset{\frown}{AB}$. Let $NB = a$ and $NA = b$. Is it possible for a, b, and c to all be distinct positive integers? Proposed by Stanley Rabinowitz [1].



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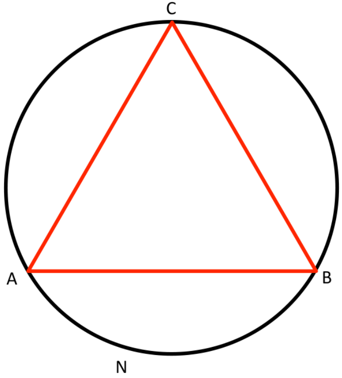

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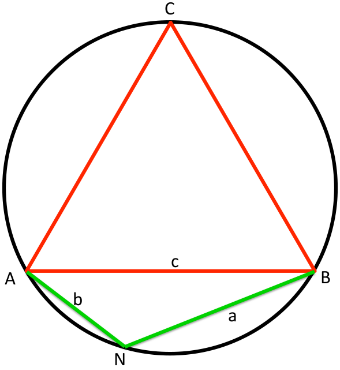

[Figure 3]

Yes!

For example, when $c = 7$, it can be shown that a solution occurs when $a = 3$ and $b = 5$.

- Are there more solutions?
- Can we find all solutions?
- Does this geometric problem connect to other areas of mathematics?

YES, again!

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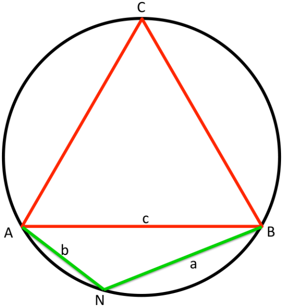
Algebraic Representation of a, b, and c

Solution. We want to find distinct integers a, b, and c.



- It follows that $m\angle ANB = 120°$.
- By the Law of Cosines,

$$c^2 = a^2 + b^2 - 2ab\cos(120°).$$

- Since $\cos(120°) = -\frac{1}{2}$,

$$c^2 = a^2 + b^2 + ab.$$

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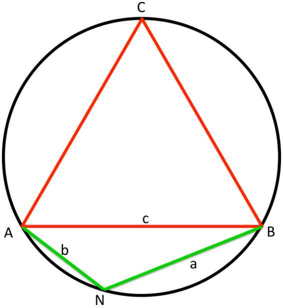

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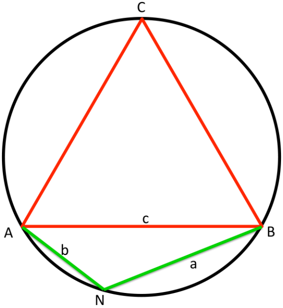

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- We can show that positive integers a, b, and c are distinct using $c^2 = a^2 + b^2 + ab$.
- It is trivial that $c > a$ and $c > b$.
- Suppose $a = b$. Then

$$\begin{aligned} c^2 &= a^2 + b^2 + ab \\ c^2 &= 3a^2 \\ \frac{c}{a} &= \sqrt{3} \end{aligned}$$

  Because $\sqrt{3}$ is not rational today, we have a contradiction and $a \neq b$!
- Thus our a, b, and c are always distinct!

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- We can manipulate our equation, such that,

$$\begin{aligned} 4c^2 &= 4a^2 + 4b^2 + 4ab \\ 4c^2 &= 4a^2 + 4ab + b^2 + 3b^2 \\ (2c)^2 &= (2a+b)^2 + 3b^2. \end{aligned}$$

- Next, we let $z = 2c$, $x = 2a + b$, and $y = b$.
- This substitution yields

$$z^2 = x^2 + 3y^2.$$

- Note since $z = 2c$, z will always be even.

A Little Algebra

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- This is a very common equation that Euler studied intensely.
- It is classified as a homogeneous Diophantine Equation.
- This implies that for any positive integer s, sx, sy, and sz will also be a solution.
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Lemma

If $z^2 = x^2 + 3y^2$, then all solutions can be characterized by only considering integers x, y, and z that are relatively prime.

Proof.

Suppose that x, y, and $z \in \mathbb{Z}^+$ and satisfy $z^2 = x^2 + 3y^2$. Let d be a common divisor of x and y. This directly implies that $d^2 | x^2$ and $d^2 | 3y^2$ and $d^2 | x^2 + 3y^2$. Therefore,

$$d^2 | z^2.$$

By unique factorization

$$d | z.$$

By further argument, we discover that if d divides any of the two, x, y, or z, then it divides the third. □

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Characterizing Solutions Continued

- Assume that g is a common divisor of the x, y, and z.
- Set $x_1 = \frac{x}{g}$, $y_1 = \frac{y}{g}$, and $z_1 = \frac{z}{g}$.
- We conclude that x_1, y_1, and z_1 will satisfy $x^2 + 3y^2 = z^2$, with pairwise $\gcd(x_1, y_1, z_1) = 1$.
- We can then call x_1, y_1, and z_1 primitive since these values are not multiples of a smaller triple.
- Since z is even and pairwise $gcd(x, y, z) = 1$, we also know that x and y must be odd.

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$$z^2 = x^2 + 3y^2$$
$$z^2 - x^2 = 3y^2$$

Factoring, we have

$$(z - x)(z + x) = 3y^2.$$

- Because the $\gcd(x, z) = 1$, we can show that $\gcd(z - x, z + x) = 1$ when z is even and x is odd.
- From Niven[2], if u and v are relatively prime positive integers whose product uv is a perfect square, then u and v are both perfect squares.
- This implies that either $z - x = 3k^2$ and $z + x = t^2$ or $z + x = 3k^2$ and $z - x = t^2$.

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Case 1 $z - x = 3k^2$ and $z + x = t^2$

Adding these two equations together yields

$$2z = 3k^2 + t^2 \text{ or } z = \frac{3k^2 + t^2}{2}$$

From this we find that

$$x = \frac{3k^2 - t^2}{2} \text{ and } y = kt.$$

Note that we must choose k and t so that $gcd(k, t) = 1$, both are odd, and $3 \not| t$.

Case 2 $z + x = 3k^2$ and $z - x = t^2$

We find that $z = \frac{3k^2 + t^2}{2}$, $x = \frac{t^2 - 3k^2}{2}$ and $y = kt$.

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Case 2 $z + x = 3k^2$ and $z - x = t^2$

We find that $z = \dfrac{3k^2 + t^2}{2}$, $x = \dfrac{t^2 - 3k^2}{2}$ and $y = kt$.

This means that all primitive solutions are categorized by

$$x = \left| \frac{3k^2 - t^2}{2} \right|,$$

$$y = kt,$$

and

$$z = \frac{3k^2 + t^2}{2},$$

where k and t are positive integers chosen so that $gcd(k, t) = 1$, both are odd, and $3 \nmid t$.

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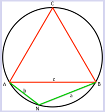

Finding a, b, and c

Recall that $z = 2c$, $x = 2a + b$, and $y = b$.

First,

$$\begin{aligned} b &= y = kt \\ b &= kt \end{aligned}$$

Then,

$$\begin{aligned} x = 2a + b &= \left|\frac{3k^2 - t^2}{2}\right| \\ a &= \left|\frac{3k^2 - t^2}{4}\right| - \frac{kt}{2} \end{aligned}$$

Finally,

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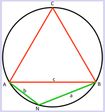

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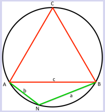

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Observe that a is not always positive. This will add a few extra conditions. Since, $a > 0$ is equivalent to

$$\left(\frac{3k^2 - t^2}{4}\right)^2 > \frac{k^2t^2}{4}$$

or

$$\begin{aligned} 9k^4 - 16k^2t^2 + t^4 &> 0 \\ (k^2 - t^2)(9k^2 - t^2) &> 0 \\ (k - t)(k + t)(3k - t)(3k + t) &> 0 \\ (k - t)(3k - t) &> 0 \end{aligned}$$

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To summarize, we have proven the following theorem.

Theorem

For every odd positive integers k and t where $3 \nmid t$, such that either $k > t$ or $3k < t$, all solutions to our triangle are of the form

$$a = s\left(\left|\frac{3k^2 - t^2}{4}\right| - \frac{kt}{2}\right)$$

$$b = s(kt)$$

and

$$c = s\left(\frac{3k^2 + t^2}{4}\right),$$

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- We have fully characterized all of the solutions.
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Conclusions

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- We found an application for integer solutions of the equation $x^2 + 3y^2 = z^2$.

Future Research

- Do further studies on equations of the form $x^2 + ny^2 = z^2$.
- For example, if our triangle is not equilateral, it can be shown:
  - $n > 0$
  - $n \neq d^2$
  - $n \equiv 0 \pmod 4$
  - $n \equiv 3 \pmod 4$
- Learn more number theory!

1. Stanley Rabinowitz. Problems for Solution No. 1245. (Fall 2011). Problem Department, *Pi Mu Epsilon Journal*.
2. Niven, I., Zuckerman, H., and Montgomery, H. (1991). *An Introduction to the Theory of Numbers*. Wiley.

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